# LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

**M.Sc.** DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – **NOVEMBER 2018**

**16/17/18PMT1MC02– REAL ANALYSIS**

Date: 27-10-2018 Dept. No. Max. : 100 Marks

Time: 01:00-04:00

**Answer all Questions.**

1. (a) State and prove the intermediate value theorem of a continuous function defined on an interval.

**(OR)**

(b) Prove that a mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y. **(5 marks)**

(c) (i) Suppose f is continuous on [a,b], $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval which contains the range of f and g is differentiable at the point f(x). If $h(t) = g(f(t)), a \leq t \leq b$, then prove that h is differentiable at x and $h'(x) = g'(f(x))f'(x)$

(ii) Suppose f is a real differentiable function on [a,b] and $f'(a) < \lambda < f'(b)$. Prove that there is a point $x \in (a, b)$ such that $f'(x) = \lambda$. **(10+5 marks)**

**(OR)**

(d) (i) If f and g are continuous real functions on [a,b] which are differentiable in (a,b), then prove that there is a point at which $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$ and hence prove the mean value theorem.

(ii)Determine all the numbers c which satisfy mean value theorem for the function $f(x) = x^3 + 2x^2 - x$ on [-1, 2]. **(10+5marks)**

2. (a) Suppose $\alpha$ increases on [a,b], $a \leq y_0 \leq b$, $\alpha$ is continuous at $y_0$, $f(y_0) = 1$ and $f(x) = 0$ if $x \neq y_0$. Prove that $f \in \Re(\alpha)$ and $\int f d\alpha = 0$.

**(OR)**

(b) If $f \in \Re(\alpha)$, then prove that $|f| \in \Re(\alpha)$ and $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha$ (5 marks)

(c) (i) Prove that $f \in \Re(\alpha)$ on [a, b] if and only if for every $\epsilon > 0$, there exists a partition **P** such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

(ii) Any monotone function f: [0, 1]→R is Riemann Integrable. Justify. **(9+6 marks)**

**(OR)**

(d) (i) Assume $\alpha$ increases monotonically and $\alpha' \in \Re$ on [a, b]. Let f be a bounded real function on [a,b]. Then prove that $f \in \Re(\alpha)$ if and only if $f\alpha' \in \Re$. Also prove that $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.

(ii) State and prove the fundamental theorem of calculus. **(9+6 marks)**

3. (a) State and prove the Cauchy criterion for uniform convergence of sequence of functions.

**(OR)**

(b) Prove that for, $f_n(x) = \frac{sinnx}{\sqrt{n}}, x\, real, n = 1,2 \ldots,$

$\lim_{n\to\infty} f_n{}'(0) \neq f'(0).$

(c) If $\{f_n\}$ is a sequence of differentiable functions on [a, b] such that $\{f_n(x_0)\}$ converges for $x_0 \in [a,b]$ and $\{f_n{}'\}$ converges uniformly on [a, b] then prove that $\{f_n\}$ converges uniformly on [a, b] to a function f and $\lim_{n\to\infty} f_n'(x) = f'(x)$.

**(OR)**

(d) State and prove the Stone-Weierstrass theorem. **(15 marks)**

4. (a) State and prove the Bessel's Inequality and hence derive the Parseval's formula.

**(OR)**

(b) ) Let $S = \{\varphi_0, \varphi_1, \varphi_2, \ldots\}$ be orthnormal on I and assume that $f \in L^2(I)$. Define two sequences of functions $\{s_n\}$and $\{t_n\}$ on I as follows: $s_n(x) = \sum_{k=0}^{\infty} c_k\varphi_k(x)$, $t_n(x) = \sum_{k=0}^{\infty} b_k\varphi_k(x)$ where $c_k = (f, \varphi_k(x)$ for k = 0, 1, 2… and $b_0$, $b_1$, $b_2$ … are arbitrary complex numbers. Then for each n, prove that$\|f - s_n\| \leq \|f - t_n\|$ **(5 marks)**

(c) (i) State and prove Riemann-Lebesgue lemma.

(ii) If $f \in L[0,2\pi]$, f is periodic with period $2\pi$ , then prove that the Fourier series generated by f converges for a given value of x if and only if for some $\delta < \pi$, $\lim_{n\to\infty} \frac{2}{\pi}\int_0^{\delta}\left(\frac{f(x+t)+f(x-t)}{2}\right)\frac{sin\left(n+\frac{1}{2}\right)t}{t}\,dt$exists and in this case this limit is the sum of the series. **(7+8 marks)**

**(OR)**

(d) (i) If $f \in L[0,2\pi]$, f is periodic with period $2\pi$ and $\{s_n\}$ is a sequence of partial sums of Fourier series generated by f, $s_n = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k coskx + b_k sinkx), n = 1,2 \ldots$ then prove that $s_n(x) = \frac{2}{\pi}\int_0^{\pi}\frac{f(x+t)+f(x-t)}{2}D_n(t)dt.$

(ii) State and prove Fejer's theorem. **(7+8 marks)**

5. (a) Prove that $\Omega$, the set of all invertible linear operators on $R^n$, is an open subset of $L(R^n)$.

**(OR)**

(b) Suppose X is a complete metric space and ϕ is a contraction of X into X. Prove that there exist one and only one x∈X such that ϕ (x) = x. **(5 marks)**

(c) State and prove the inverse function theorem.

**(OR)**

(d) State and prove the implicit function theorem. **(15 marks)**

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